4-3 Notes: Definite Integral
Calculus AB

Name ___________________________

Definition of the Area of a Region in the Plane

Let f be continuous and non-negative (above the x-axis) on [a, b]. The area of the region bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x \text{ where } x_{i-1} \le c_i \le x_i \text{ and } \Delta x = \frac{b-a}{n}$$

(this is basically an infinite Riemann Sum)

We have used the limit of a sum to define the area of a plane region. (It can also be used to find volumes, surface area, average value, arc length, work, centroids, to name only a few.) The limit of a sum process can be cumbersome even with simple functions. The definite integral will now be defined in terms of the limit of a sum (the integration process is much easier than finding the limit of the sum).

Remember that the indefinite integral is really a family of curves. However, since the definite integral is a limit of a sum, it will therefore be a number, not a family of functions!

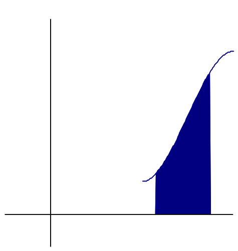
Informal Definition of Definite Integral

Suppose f is continuous for $a \le x \le b$. The definite integral from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \quad \text{OR} \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

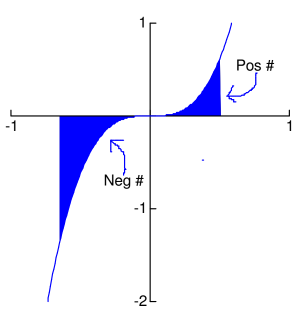
$f(x)$ is called the integrand, a and b are called the limits of integration.

Up to this point, we have been looking a functions that were non-negative. This was because we were restricting our work with Riemann sums to finding area. We can integrate functions that go below the x-axis as long as we note:



When $f(x)$ is positive of [a, b], (where a<b) the definite integral represents the area between the x-axis, the function and the lines $x = a$ and

$$\int_a^b f(x)\,dx = shaded\ area$$



When $f(x)$ is positive for some x-values and negative for others (goes below the x-axis) on [a, b], (where a<b)

$\int_a^b f(x)\,dx$ is the sum of the area above the x-axis counted positively and the area below the x-axis counted negatively.

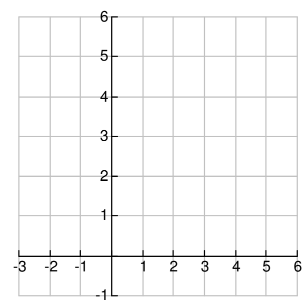
The Definite Integral as the Area of a Region

If f is continuous and non-negative on the closed interval [a, b], then the area of the region bounded by the graph of f , the x-axis and the vertical lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$

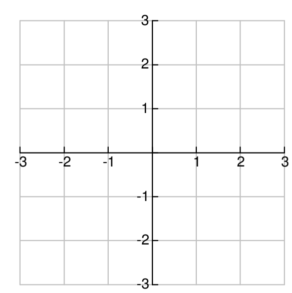
- The definite integral gives an area for a region only if the function is above the x-axis for the entire interval
- Otherwise, the integral will be positive when the area above the x-axis > the area below
- OR the integral will be negative when area above the x-axis < area below
- OR the integral will be zero when the area above the x-axis = area below

Sketch the shaded region whose area is represented by the following integrals, then use geometry to find the value of the integral

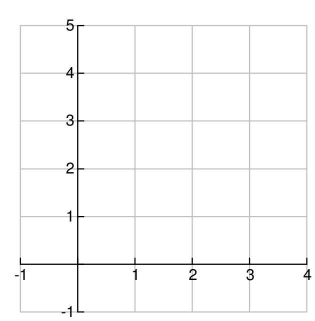
Example 1) $\int_0^3 (x+2)\,dx$



Example 2) $\int_{-2}^2 \sqrt{4-x^2}\,dx$



Example 3) $\int_1^3 4\,dx$



Continuity Implies Integrability

If a function is continuous on the closed interval [a, b], then it can be integrated on [a, b].

- From 2.1 we found that differentiability implied continuity
- Therefore, differentiability implies Integrability
- Beware : continuity does not imply differentiability and integrability does not imply continuity

Properties of Definite Integrals

1) If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$

2) If f is integrable on [a, b], then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

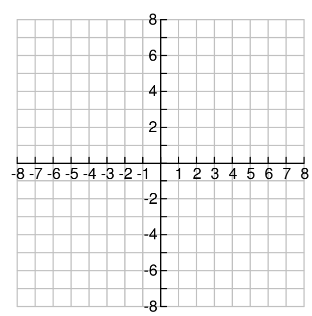
4) If f and g are integrable on [a, b] and k is a constant then the functions of $k\,f$ and $f \pm g$ are integrable on [a, b] and

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

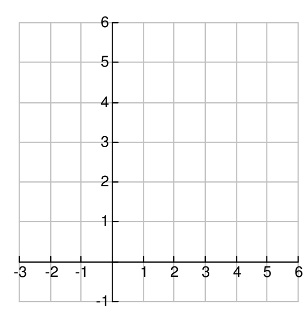
$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Sketch the region whose area is given by the definite integral, then use geometry to evaluate the integral.

Example 4) $\int_{-3}^{2} (2x)\,dx =$



Example 5) $\int_0^4 \frac{x}{2}\,dx =$



Example 6) Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find the following:

a) $\int_0^7 f(x)\,dx$

b) $\int_5^0 f(x)\,dx$

c) $\int_5^5 f(x)\,dx$

d) $\int_0^5 3f(x)\,dx$